

ARITHMETIC IDENTITIES AND CONGRUENCES FOR PARTITION TRIPLES WITH 3-CORES

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ABSTRACT. Let $B_3(n)$ denote the number of partition triples of n where each partition is 3-core. With the help of generating function manipulations, we find several infinite families of arithmetic identities and congruences for $B_3(n)$. Moreover, let $\omega(n)$ denote the number of representations of a nonnegative integer n in the form $x_1^2 + x_2^2 + x_3^2 + 3y_1^2 + 3y_2^2 + 3y_3^2$ with $x_1, x_2, x_3, y_1, y_2, y_3 \in \mathbb{Z}$. We find three arithmetic relations between $B_3(n)$ and $\omega(n)$, such as $\omega(6n+5) = 4B_3(6n+4)$.

1. INTRODUCTION

A partition of a positive integer n is any nonincreasing sequence of positive integers whose sum is n . For example, $7 = 4 + 2 + 1$ and $\lambda = \{4, 2, 1\}$ is a partition of 7. A partition λ of n is said to be a t -core if it has no hook numbers that are multiples of t . We denote by $a_t(n)$ the number of partitions of n that are t -cores. For convenience, we use the following notation

$$(a; q)_\infty = \prod_{n=1}^{\infty} (1 - aq^n), \quad \text{and} \quad f_k = (q^k; q^k)_\infty.$$

From [11, Eq. (2.1)], the generating function of $a_t(n)$ is given by

$$\sum_{n=0}^{\infty} a_t(n) q^n = \frac{f_t^t}{f_1}.$$

In particular, for $t = 3$, we have

$$\sum_{n=0}^{\infty} a_3(n) q^n = \frac{f_3^3}{f_1}.$$

A partition k -tuple $(\lambda_1, \lambda_2, \dots, \lambda_k)$ of n is a k -tuple of partitions $\lambda_1, \lambda_2, \dots, \lambda_k$ such that the sum of all the parts equals n . For example, let $\lambda_1 = \{3, 1\}$, $\lambda_2 = \{1, 1\}$, $\lambda_3 = \{1\}$. Then $(\lambda_1, \lambda_2, \lambda_3)$ is a partition triple of 7 since $3 + 1 + 1 + 1 + 1 = 7$. A partition k -tuple of n with t -cores is a partition k -tuple $(\lambda_1, \lambda_2, \dots, \lambda_k)$ of n where each λ_i is t -core for $i = 1, 2, \dots, k$.

Let $A_t(n)$ (resp. $B_t(n)$) denote the number of bipartitions (resp. partition triples) of n with t -cores. Then the generating functions for $A_t(n)$ and $B_t(n)$ are given by

$$\sum_{n=0}^{\infty} A_t(n) q^n = \frac{f_t^{2t}}{f_1^2}$$

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and

$$\sum_{n=0}^{\infty} B_t(n)q^n = \frac{f_t^{3t}}{f_1^3} \quad (1.1)$$

respectively.

In 1996, Granville and Ono [9] found that

$$a_3(n) = d_{1,3}(3n+1) - d_{2,3}(3n+1), \quad (1.2)$$

where $d_{r,3}(n)$ denote the number of divisors of n congruent to r modulo 3. Their proof is based on the theory of modular forms.

Baruah and Berndt [3] showed that for any nonnegative integer n ,

$$a_3(4n+1) = a_3(n).$$

In 2009, Hirschhorn and Sellers [12] provided an elementary proof of (1.2) and as corollaries, they proved some arithmetic identities. For example, let $p \equiv 2 \pmod{3}$ be prime and let k be a positive even integer. Then, for all $n \geq 0$,

$$a_3\left(p^k n + \frac{p^k - 1}{3}\right) = a_3(n).$$

Let $u(n)$ denote the number of representations of a nonnegative integer n in the form $x^2 + 3y^2$ with $x, y \in \mathbb{Z}$. By using Ramanujan's theta function identities, Baruah and Nath [4] proved that

$$u(12n+4) = 6a_3(n).$$

In 2014, Lin [13] discovered some arithmetic identities about $A_3(n)$. For example, he proved that $A_3(8n+6) = 7A_3(2n+1)$. Let $v(n)$ denote the number of representations of a nonnegative integer n in the form $x_1^2 + x_2^2 + 3y_1^2 + 3y_2^2$ with $x_1, x_2, y_1, y_2 \in \mathbb{Z}$. Lin showed that

$$v(6n+5) = 12A_3(2n+1). \quad (1.3)$$

Again, Baruah and Nath [2] generalized (1.3) and established three infinite families of arithmetic identities involving $A_3(n)$. For example, for any integer $k \geq 1$,

$$A_3\left(2^{2k+2}n + \frac{2(2^{2k+2} - 1)}{3}\right) = \frac{2^{2k+2} - 1}{3} \cdot A_3(4n+2) - \frac{2^{2k+2} - 4}{3} \cdot A_3(n).$$

For more results and details about $a_3(n)$ and $A_3(n)$, see [1, 2, 3, 4, 12, 13].

Motivated by their work, we study the arithmetic properties of partition triples with 3-cores. By using some identities of q series, we prove some analogous results. We will show that

$$B_3(4n+1) = 3B_3(2n), \quad B_3(3n+2) = 9B_3(n), \quad \text{and}$$

$$B_3(4n+3) = 3B_3(2n+1) + 4B_3(n).$$

From these relations we deduce three infinite families of arithmetic identities as well as some Ramanujan-type congruences involving $B_3(n)$. For example, we prove two infinite families of congruences for $B_3(n)$: for $k \geq 1$ and all $n \geq 0$,

$$\begin{aligned} B_3(2^{k+1}n + 2^k - 1) &\equiv 0 \pmod{\frac{4^{k+1} + (-1)^k}{5}}, \\ B_3(3^k n + 3^k - 1) &\equiv 0 \pmod{3^{2k}}, \\ B_3(3^k n + 2 \cdot 3^{k-1} - 1) &\equiv 0 \pmod{3^{2k-1}}. \end{aligned}$$

We will also prove that

$$\sum_{n=0}^{\infty} B_3(6n+4)q^n = 24 \frac{f_2^8 f_3^3}{f_1^5},$$

from which we deduce the following two Ramanujan-type congruences:

$$B_3(30n+10) \equiv B_3(30n+28) \equiv 0 \pmod{120}.$$

Furthermore, let $\omega(n)$ denote the number of representations of a nonnegative integer n in the form

$$n = x_1^2 + x_2^2 + x_3^2 + 3y_1^2 + 3y_2^2 + 3y_3^2, \quad x_1, x_2, x_3, y_1, y_2, y_3 \in \mathbb{Z}.$$

We find some interesting arithmetic relations between $\omega(n)$ and $B_3(n)$:

$$\omega(6n+5) = 4B_3(6n+4),$$

$$\omega(12n+2) = 12B_3(6n),$$

$$\omega(12n+10) = 6B_3(6n+4).$$

In the final section, we introduce a unified notation $A_3^{(k)}(n)$ to denote the number of partition k -tuples of n wherein each partition is 3-core. We propose two open questions about that whether we can find some analogous results about $A_3^{(k)}(n)$ for all positive integer k or not. This will lead to researches in the future.

2. MAIN RESULTS AND PROOFS

Setting $t = 3$ in (1.1), we obtain that

$$\sum_{n=0}^{\infty} B_3(n)q^n = \frac{f_3^9}{f_1^3}. \quad (2.1)$$

The following 2-dissection identities will be important in our proofs.

Lemma 2.1. *We have*

$$\frac{f_3^3}{f_1} = \frac{f_4^3 f_6^2}{f_2^2 f_{12}} + q \frac{f_{12}^3}{f_4}, \quad (2.2)$$

$$\frac{f_3}{f_1^3} = \frac{f_4^6 f_6^3}{f_2^9 f_{12}^2} + 3q \frac{f_4^2 f_6 f_{12}^2}{f_2^7}, \quad (2.3)$$

$$\frac{f_1^3}{f_3} = \frac{f_4^3}{f_{12}} - 3q \frac{f_2^2 f_{12}^3}{f_4 f_6^2}, \quad (2.4)$$

$$\frac{f_1}{f_3^3} = \frac{f_2 f_4^2 f_{12}^2}{f_6^7} - q \frac{f_2^3 f_{12}^6}{f_4^2 f_6^9}, \quad (2.5)$$

$$\frac{1}{f_1^4} = \frac{f_4^{14}}{f_2^{14} f_8^4} + 4q \frac{f_4^2 f_8^4}{f_2^{10}}, \quad (2.6)$$

$$\frac{1}{f_1^8} = \frac{f_4^{28}}{f_2^{28} f_8^8} + 8q \frac{f_4^{16}}{f_2^{24}} + 16q^2 \frac{f_4^4 f_8^8}{f_2^{20}}. \quad (2.7)$$

Proof. For the proofs of (2.2)–(2.3), see [14, Eq. (3.75) and Eq. (3.38)]. The proofs of (2.4)–(2.5) can be found in [11]. For a proof of (2.6), see [14, Eq. (2.11)]. (2.7) follows by squaring both sides of (2.6). \square

Lemma 2.2. *We have*

$$\frac{f_2^8 f_3^4}{f_1^4} = \frac{f_2^3 f_3^9}{f_1^3 f_6} + q f_6^8, \quad (2.8)$$

$$\frac{f_2^5 f_3^4 f_6}{f_1^4} = \frac{f_3^9}{f_1^3} + q \frac{f_6^9}{f_2^3}. \quad (2.9)$$

Proof. For convenience, we introduce the following notation

$$[x; q]_\infty = (x; q)_\infty (q/x; q)_\infty, \\ [a_1, \dots, a_n; q]_\infty = \prod_{i=1}^n [a_i; q]_\infty.$$

Multiplying both sides of (2.8) by $f_1^4 f_6$, we know (2.8) is equivalent to

$$f_2^8 f_3^4 f_6 = f_1 f_2^3 f_3^9 + q f_1^4 f_6^9. \quad (2.10)$$

Note that

$$f_1 = [q, q^2; q^6]_\infty (q^3; q^6)_\infty (q^6; q^6)_\infty, \quad f_2 = [q^2; q^6]_\infty (q^6; q^6)_\infty, \\ f_3 = (q^3; q^6)_\infty (q^6; q^6)_\infty, \quad f_6 = (q^6; q^6)_\infty.$$

Substituting these expressions into (2.10) and simplifying, we know (2.10) is equivalent to

$$[q^2; q^6]_\infty^4 = [q, q^3, q^3, q^3; q^6]_\infty + q [q; q^6]_\infty^4. \quad (2.11)$$

From [8, Exercise 2.16, p. 61], we know

$$[x\lambda, x/\lambda, \mu v, \mu/v; q]_\infty = [xv, x/v, \lambda\mu, \mu/\lambda; q]_\infty + \frac{\mu}{\lambda} [x\mu, x/\mu, \lambda v, \lambda/v; q]_\infty. \quad (2.12)$$

Taking $(x, \lambda, \mu, v, q) \rightarrow (q^3, q, q^2, 1, q^6)$ in (2.12), we have

$$[q^4, q^2, q^2, q^2; q^6]_\infty = [q, q^3, q^3, q^3; q^6]_\infty + q [q^5, q, q, q; q^6]_\infty.$$

Hence (2.11) holds and we complete our proof of (2.8).

From (2.2) and (2.3), we obtain that

$$\frac{f_3^4}{f_1^4} = \frac{f_3^3}{f_1} \cdot \frac{f_3}{f_1} = \frac{f_4^9 f_6^5}{f_2^{11} f_{12}^3} + 3q^2 \frac{f_4 f_6 f_{12}^5}{f_2^7} + 4q \frac{f_4^5 f_6^3 f_{12}}{f_2^9}. \quad (2.13)$$

Multiplying both sides by f_2^8 , we get

$$\frac{f_2^8 f_3^4}{f_1^4} = \frac{f_4^9 f_6^5}{f_2^3 f_{12}^3} + 4q \frac{f_4^5 f_6^3 f_{12}}{f_2} + 3q^2 f_2 f_4 f_6 f_{12}^5. \quad (2.14)$$

Applying (2.2), we obtain that

$$\frac{f_2^3 f_3^9}{f_1^3 f_6} = \frac{f_2^3}{f_6} \cdot \left(\frac{f_3}{f_1} \right)^3 = \frac{f_4^9 f_6^5}{f_2^3 f_{12}^3} + 3q \frac{f_4^5 f_6^3 f_{12}}{f_2} + 3q^2 f_2 f_4 f_6 f_{12}^5 + q^3 \frac{f_2^3 f_{12}^9}{f_4^3 f_6}. \quad (2.15)$$

Substituting (2.14) and (2.15) into (2.8), we deduce that

$$\frac{f_4^5 f_6^3 f_{12}}{f_2} = f_6^8 + q^2 \frac{f_2^3 f_{12}^9}{f_4^3 f_6}.$$

Replacing q^2 by q and then multiplying both sides by $\frac{f_3}{f_1^3}$, we obtain (2.9). \square

Theorem 2.1. *For any integer $k \geq 1$, we have*

$$B_3(2^{k+1}n + 2^k - 1) = \frac{2^{2k+2} + (-1)^k}{5} \cdot B_3(2n), \quad (2.16)$$

$$B_3(2^{k+1}n + 2^{k+1} - 1) = \frac{2^{2k+2} + (-1)^k}{5} \cdot B_3(2n + 1) + \frac{2^{2k+2} - 4(-1)^k}{5} \cdot B_3(n). \quad (2.17)$$

Proof. Substituting (2.2) into (2.1), we obtain that

$$\begin{aligned} \sum_{n=0}^{\infty} B_3(n)q^n &= \left(\frac{f_4^3 f_6^2}{f_2^2 f_{12}} + q \frac{f_{12}^3}{f_4} \right)^3 \\ &= \left(\frac{f_4^9 f_6^6}{f_2^6 f_{12}^3} + 3q^2 \frac{f_4 f_6^2 f_{12}^5}{f_2^2} \right) + q \left(3 \frac{f_4^5 f_6^4 f_{12}}{f_2^4} + q^2 \frac{f_{12}^9}{f_4^3} \right). \end{aligned}$$

Extracting the terms involving q^{2n} and q^{2n+1} , respectively, we get

$$\sum_{n=0}^{\infty} B_3(2n)q^n = \frac{f_2^9 f_3^6}{f_1^6 f_6^3} + 3q \frac{f_2 f_3^2 f_6^5}{f_1^2}, \quad (2.18)$$

and

$$\sum_{n=0}^{\infty} B_3(2n+1)q^n = 3 \frac{f_2^5 f_3^4 f_6}{f_1^4} + q \frac{f_6^9}{f_2^3}. \quad (2.19)$$

By (2.9), we deduce that

$$\begin{aligned} \sum_{n=0}^{\infty} B_3(2n+1)q^n &= 3 \frac{f_3^9}{f_1^3} + 4q \frac{f_6^9}{f_2^3} \\ &= 3 \sum_{n=0}^{\infty} B_3(n)q^n + 4 \sum_{n=0}^{\infty} B_3(n)q^{2n+1}. \end{aligned}$$

Equating the coefficients of q^{2n} and q^{2n+1} on both sides, respectively, we obtain

$$B_3(4n+1) = 3B_3(2n) \quad (2.20)$$

and

$$B_3(4n+3) = 3B_3(2n+1) + 4B_3(n). \quad (2.21)$$

We are now able to prove (2.16)–(2.17). Note that (2.20) and (2.21) are (2.16) and (2.17), respectively, for $k = 1$. Now we prove (2.16). Replacing n by $2n$ in (2.21), we have

$$B_3(8n+3) = 3B_3(4n+1) + 4B_3(2n).$$

By (2.20), this implies

$$B_3(8n+3) = 13B_3(2n),$$

which is (2.16) for $k = 2$. Now the proof of (2.16) follows by mathematical induction.

Next, replacing n by $2n+1$ in (2.21), we have

$$B_3(8n+7) = 3B_3(4n+3) + 4B_3(2n+1).$$

Employing (2.21) in the above, we deduce that

$$B_3(8n+7) = 13B_3(2n+1) + 12B_3(n),$$

which is (2.17) for $k = 2$. Now the proof of (2.17) can be completed by mathematical induction. \square

Corollary 2.1. *For any integer $k \geq 1$, we have*

$$B_3(2^{k+1}n + 2^k - 1) \equiv 0 \pmod{\frac{2^{2k+2} + (-1)^k}{5}}.$$

Recall that the general Ramanujan's theta function $f(a, b)$ is defined by

$$f(a, b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \quad |ab| < 1.$$

As some special cases, we have (see [6], for example)

$$\varphi(q) := f(q, q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q; q^2)_{\infty}^2 (q^2; q^2)_{\infty}, \quad (2.22)$$

$$\psi(q) := f(q, q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2; q^2)_{\infty}^2}{(q; q)_{\infty}^2} \quad (2.23)$$

and

$$f(-q) := f(-q, -q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q; q)_{\infty}. \quad (2.24)$$

Lemma 2.3. *We have*

$$(q; q)_{\infty}^3 = P(q^3) - 3q(q^9; q^9)^3, \quad (2.25)$$

where

$$P(q) = \sum_{m=-\infty}^{\infty} (-1)^m (6m+1) q^{m(3m+1)/2} = f(-q) \varphi(q) \varphi(q^3) + 4q f(-q) \psi(q^2) \psi(q^6). \quad (2.26)$$

Proof. By Jacobi's identity (see [6, Theorem 1.3.9]), we have

$$(q; q)_{\infty}^3 = \sum_{n=0}^{\infty} (-1)^n (2n+1) q^{n(n+1)/2}.$$

Note that $\frac{n(n+1)}{2} \equiv 0 \pmod{3}$ if and only if $n \equiv 0 \pmod{3}$ or $n \equiv 2 \pmod{3}$. And $\frac{n(n+1)}{2} \equiv 1 \pmod{3}$ if and only if $n \equiv 1 \pmod{3}$. Hence we have the following 3-dissection identity

$$(q; q)_{\infty}^3 = P(q^3) + qR(q^3).$$

We have

$$\begin{aligned} P(q^3) &= \sum_{m=0}^{\infty} (-1)^{3m} (6m+1) q^{3m(3m+1)/2} + \sum_{m=0}^{\infty} (-1)^{3m+2} (6m+5) q^{(3m+2)(3m+3)/2} \\ &= \sum_{m=0}^{\infty} (-1)^m (6m+1) q^{3m(3m+1)/2} + \sum_{m=-\infty}^{-1} (-1)^m (6m+1) q^{3m(3m+1)/2} \\ &= \sum_{m=-\infty}^{\infty} (-1)^m (6m+1) q^{3m(3m+1)/2}. \end{aligned}$$

Replacing q^3 by q , we obtain

$$P(q) = \sum_{m=-\infty}^{\infty} (-1)^m (6m+1) q^{m(3m+1)/2}.$$

From [10] we know that

$$P(q) = (q; q)_{\infty} \left(1 + 6 \sum_{n \geq 0} \left(\frac{q^{3n+1}}{1 - q^{3n+1}} - \frac{q^{3n+2}}{1 - q^{3n+2}} \right) \right).$$

From [6, Theorem 3.7.9] we have

$$1 + 6 \sum_{n \geq 0} \left(\frac{q^{3n+1}}{1 - q^{3n+1}} - \frac{q^{3n+2}}{1 - q^{3n+2}} \right) = \varphi(q) \varphi(q^3) + 4q \psi(q^2) \psi(q^6).$$

Hence (2.26) is proved.

Again, we have

$$qR(q^3) = \sum_{m=0}^{\infty} (-1)^{3m+1} (6m+3) q^{(3m+1)(3m+2)/2}.$$

Dividing both sides by q and replacing q^3 by q , we deduce that

$$R(q) = -3 \sum_{m=0}^{\infty} (-1)^m (2m+1) q^{3m(m+1)/2} = -3(q^3; q^3)_{\infty}^3.$$

□

Lemma 2.4. *We have*

$$P(q)^3 - 27q(q^3; q^3)_{\infty}^9 = \frac{(q; q)_{\infty}^{12}}{(q^3; q^3)_{\infty}^3}, \quad (2.27)$$

and

$$\frac{1}{(q; q)_{\infty}^3} = \frac{(q^9; q^9)_{\infty}^3}{(q^3; q^3)_{\infty}^{12}} \left(P(q^3)^2 + 3qP(q^3)(q^9; q^9)_{\infty}^3 + 9q^2(q^9; q^9)_{\infty}^6 \right). \quad (2.28)$$

Proof. Let $\omega = e^{2\pi i/3}$. On the one hand, by Lemma 2.3, we have

$$(q; q)_{\infty}^3 (\omega q; \omega q)_{\infty}^3 (\omega^2 q; \omega^2 q)_{\infty}^3 = \prod_{k=0}^2 \left(P(q^3) - 3\omega^k q f_9^3 \right) = P(q^3)^3 - 27q^3 f_9^9.$$

On the other hand, by definition we have

$$\begin{aligned}
& (q; q)_\infty^3 (\omega q; \omega q)_\infty^3 (\omega^2 q; \omega^2 q)_\infty^3 \\
&= \prod_{n=1}^{\infty} (1 - q^n)^3 (1 - \omega^n q^n)^3 (1 - \omega^{2n} q^n)^3 \\
&= \left(\prod_{3|n} (1 - q^n) (1 - \omega^n q^n) (1 - \omega^{2n} q^n) \right)^3 \cdot \prod_{3 \nmid n} (1 - q^{3n})^3 \\
&= \prod_{n=1}^{\infty} (1 - q^{3n})^9 \cdot \prod_{n=1}^{\infty} (1 - q^{3n})^3 / \prod_{n=1}^{\infty} (1 - q^{9n})^3 \\
&= \frac{(q^3; q^3)_\infty^{12}}{(q^9; q^9)_\infty^3}.
\end{aligned}$$

Hence we deduce that

$$P(q^3)^3 - 27q^3 f_9^9 = \frac{(q^3; q^3)_\infty^{12}}{(q^9; q^9)_\infty^3}.$$

Replacing q^3 by q we obtain (2.27).

By (2.25) we have

$$\begin{aligned}
\frac{1}{(q; q)_\infty^3} &= \frac{\prod_{k=1}^2 (P(q^3) - 3\omega^k q f_9^3)}{\prod_{k=0}^3 (P(q^3) - 3\omega^k q f_9^3)} \\
&= \frac{1}{P(q^3)^3 - 27q^3 f_9^9} \cdot \left(P(q^3)^2 + 3qP(q^3)f_9^3 + 9q^2 f_9^6 \right) \\
&= \frac{f_9^3}{f_3^{12}} \left(P(q^3)^2 + 3qP(q^3)f_9^3 + 9q^2 f_9^6 \right),
\end{aligned}$$

where the last equality follows from (2.27). \square

Theorem 2.2. *We have*

$$\sum_{n=0}^{\infty} B_3(3n)q^n = P(q)^2 \frac{(q^3; q^3)_\infty^3}{(q; q)_\infty^3}, \quad (2.29)$$

$$\sum_{n=0}^{\infty} B_3(3n+1)q^n = 3P(q) \frac{(q^3; q^3)_\infty^6}{(q; q)_\infty^3}, \quad (2.30)$$

$$\sum_{n=0}^{\infty} B_3(3n+2)q^n = 9 \frac{(q^3; q^3)_\infty^9}{(q; q)_\infty^3}, \quad (2.31)$$

$$(2.32)$$

Proof. By (2.1) and (2.28) we have

$$\sum_{n=0}^{\infty} B_3(n)q^n = \frac{(q^3; q^3)_\infty^9}{(q; q)_\infty^3} = \frac{(q^9; q^9)_\infty^3}{(q^3; q^3)_\infty^3} \left(P(q^3)^2 + 3qP(q^3)f_9^3 + 9q^2 f_9^6 \right).$$

Extracting the terms involving q^{3n} , q^{3n+1} and q^{3n+2} , respectively, we get the desired results. \square

Theorem 2.3. *For any integer $k \geq 1$, we have*

$$B_3(3^k n + 3^k - 1) = 3^{2k} B_3(n).$$

Proof. From (2.1) and (2.31) we deduce that

$$B_3(3n + 2) = 9B_3(n). \quad (2.33)$$

This proves the theorem for $k = 1$. Replacing n by $3n + 2$ in (2.33), we deduce that

$$B_3(9n + 8) = 3^2 B_3(3n + 2) = 3^4 B_3(n),$$

which proves the theorem for $k = 2$. The theorem now follows by induction on k . \square

Corollary 2.2. *For any integer $k \geq 1$, we have*

$$\begin{aligned} B_3(3^k n + 3^k - 1) &\equiv 0 \pmod{3^{2k}}, \\ B_3(3^k n + 2 \cdot 3^{k-1} - 1) &\equiv 0 \pmod{3^{2k-1}}. \end{aligned}$$

Proof. The first congruence follows immediately from Theorem 2.3.

By (2.31), we know that $B_3(3n + 1) \equiv 0 \pmod{3}$, and this proves the second congruence for $k = 1$. For $k \geq 2$, by Theorem 2.3, we have

$$B_3(3^k n + 2 \cdot 3^{k-1} - 1) = B_3(3^{k-1}(3n + 1) + 3^{k-1} - 1) = 3^{2k-2} B_3(3n + 1) \equiv 0 \pmod{3^{2k-1}}.$$

\square

Theorem 2.4. *We have*

$$\sum_{n=0}^{\infty} B_3(6n) q^n = \frac{f_2^{10} f_3^9}{f_1^7 f_6^6} + 16q \frac{f_2^7 f_6^3}{f_1^4} + 27q \frac{f_2^2 f_3^5 f_6^2}{f_1^3} \quad (2.34)$$

$$\sum_{n=0}^{\infty} B_3(6n + 4) q^n = 24 \frac{f_2^8 f_3^3}{f_1^5}. \quad (2.35)$$

Proof. From (2.22) and (2.23), it is not hard to see that

$$\varphi(q) = \frac{f_2^5}{f_1^2 f_4^2}, \quad \psi(q) = \frac{f_2^2}{f_1}. \quad (2.36)$$

By (2.26), we have

$$P(q) = \frac{f_2^5 f_6^5}{f_1 f_3^2 f_4^2 f_{12}^2} + 4q \frac{f_1 f_4^2 f_{12}^2}{f_2 f_6}. \quad (2.37)$$

Substituting (2.37) into (2.29), we obtain

$$\sum_{n=0}^{\infty} B_3(3n) q^n = \left(\frac{f_2^{10} f_6^{10}}{f_1^5 f_3 f_4^4 f_{12}^4} + 16q^2 \frac{f_3^3 f_4^4 f_{12}^4}{f_1 f_2^2 f_6^2} \right) + 8q \frac{f_2^4 f_3 f_6^4}{f_1^3}. \quad (2.38)$$

By (2.3) and (2.5), we have

$$\frac{1}{f_1^5 f_3} = \left(\frac{f_3}{f_1^3} \right)^2 \cdot \frac{f_1}{f_3^3} = \left(\frac{f_4^{14}}{f_2^{17} f_6 f_{12}^2} + 3q^2 \frac{f_4^6 f_{12}^6}{f_6^5 f_2^{13}} \right) + q \left(5 \frac{f_4^{10} f_{12}^2}{f_2^{15} f_6^3} - 9q^2 \frac{f_4^2 f_{12}^{10}}{f_6^7 f_2^{11}} \right). \quad (2.39)$$

Now substituting (2.2), (2.3) and (2.39) into (2.38), and then extracting the terms involving q^{2n} , we obtain

$$\sum_{n=0}^{\infty} B_3(6n) q^{2n} = \frac{f_6^9 f_4^{10}}{f_2^7 f_{12}^6} + 16q^2 \frac{f_4^7 f_{12}^3}{f_2^4} + 27q^2 \frac{f_4^2 f_6^5 f_{12}^2}{f_2^3}.$$

Replacing q^2 by q we prove (2.34).

Similarly, substituting (2.37) into (2.30), we obtain

$$\sum_{n=0}^{\infty} B_3(3n+1)q^n = 3 \frac{f_2^5 f_3^4 f_6^5}{f_1^4 f_4^2 f_{12}^2} + 12q \frac{f_3^6 f_4^2 f_{12}^2}{f_1^2 f_2 f_6}. \quad (2.40)$$

By (2.2), we deduce that

$$\frac{f_3^6}{f_1^2} = \left(\frac{f_3^3}{f_1} \right)^2 = \frac{f_4^6 f_6^4}{f_2^4 f_{12}^2} + 2q \frac{f_4^2 f_6^2 f_{12}^2}{f_2^2} + q^2 \frac{f_{12}^6}{f_4^2}.$$

Substituting this identity and (2.13) into (2.40), and extracting the terms involving q^{2n+1} , we obtain

$$\sum_{n=0}^{\infty} B_3(6n+4)q^{2n+1} = 12q \left(\frac{f_4^3 f_6^8}{f_2^4 f_{12}} + \frac{f_4^8 f_6^3}{f_2^5} + q^2 \frac{f_{12}^8}{f_2 f_6} \right).$$

Dividing both sides by q and replacing q^2 by q , and applying (2.8) we obtain that

$$\sum_{n=0}^{\infty} B_3(6n+4)q^n = \frac{12}{f_1 f_3} \left(\frac{f_2^3 f_3^9}{f_1^3 f_6} + q f_6^8 \right) + 12 \frac{f_2^8 f_3^3}{f_1^5} = 24 \frac{f_2^8 f_3^3}{f_1^5}.$$

□

Corollary 2.3. *For any integer $n \geq 0$, we have*

$$B_3(6n+4) \equiv 0 \pmod{24}.$$

Proof. This follows from (2.35). □

Theorem 2.5. *For any integer $n \geq 0$, we have*

$$B_3(30n+10) \equiv B_3(30n+28) \equiv 0 \pmod{120}.$$

Proof. Note that for any prime $p \geq 3$, we have

$$\binom{p}{k} = \frac{p}{k} \cdot \binom{p-1}{k-1} \equiv 0 \pmod{p}, \quad 1 \leq k \leq p-1.$$

By the binomial theorem, we have

$$(1-x)^p = 1 - px + \cdots + px^{p-1} - x^p \equiv 1 - x^p \pmod{p}.$$

Hence for any integer $a \geq 1$, we have

$$f_a^p \equiv \prod_{n=1}^{\infty} (1 - q^{an})^p \equiv \prod_{n=1}^{\infty} (1 - q^{apn}) = f_{ap} \pmod{p}.$$

From (2.35) we have

$$\sum_{n=0}^{\infty} B_3(6n+4)q^n = 24 \frac{f_2^8 f_3^3}{f_1^5} \equiv 24 \frac{f_{10}}{f_5} \cdot f_2^3 f_3^3 \pmod{120}. \quad (2.41)$$

By Jacobi's identity, we have

$$f_2^3 f_3^3 = \left(\sum_{k=0}^{\infty} (-1)^k (2k+1) q^{k(k+1)} \right) \left(\sum_{l=0}^{\infty} (-1)^l (2l+1) q^{3l(l+1)/2} \right).$$

Suppose

$$f_2^3 f_3^3 = \sum_{m=0}^{\infty} c(m) q^m,$$

then

$$c(m) = \sum_{\substack{k(k+1)+3l(l+1)/2=m \\ k,l \geq 0}} (-1)^{k+l} (2k+1)(2l+1).$$

Note that

$$m = k(k+1) + \frac{3l(l+1)}{2} \Leftrightarrow 8m+5 = 2(2k+1)^2 + 3(2l+1)^2.$$

For any integer x , we have $x^2 \equiv 0, 1, 4 \pmod{5}$. If $m \equiv 1$ or $4 \pmod{5}$, then at least one of $2k+1$ or $2l+1$ must be divisible by 5. Hence we deduce that

$$c(5n+1) \equiv c(5n+4) \equiv 0 \pmod{5}.$$

By (2.41) we have

$$\sum_{n=0}^{\infty} B_3(6n+4) q^n \equiv 24 \frac{f_{10}}{f_5} \sum_{m=0}^{\infty} c(m) q^m \pmod{120}.$$

The theorem now follows by comparing the coefficients of q^{5n+r} ($r \in \{1, 4\}$) on both sides of the above identity. \square

Theorem 2.6. *Let $\omega(n)$ denote the number of representations of a nonnegative integer n in the form $x_1^2 + x_2^2 + x_3^2 + 3y_1^2 + 3y_2^2 + 3y_3^2$ with $x_1, x_2, x_3, y_1, y_2, y_3 \in \mathbb{Z}$. Then*

$$\omega(6n+5) = 4B_3(6n+4).$$

Proof. By [5, p. 49, Corollary (i)] and Jacobi triple product identity [6, Theorem 1.3.3], we can deduce that

$$\varphi(q) = \varphi(q^9) + 2q \frac{f_6^2 f_9 f_{36}}{f_3 f_{12} f_{18}}.$$

The generating function of $\omega(n)$ is given by

$$\sum_{n=0}^{\infty} \omega(n) q^n = \varphi^3(q) \varphi^3(q^3) = \varphi^3(q^3) \left(\varphi(q^9) + 2q \frac{f_6^2 f_9 f_{36}}{f_3 f_{12} f_{18}} \right)^3. \quad (2.42)$$

Extracting the terms q^{3n+2} from (2.42), dividing by q^2 , replacing q^3 by q , we obtain that

$$\sum_{n=0}^{\infty} \omega(3n+2) q^n = 12 \varphi^3(q) \varphi(q^3) \frac{f_2^4 f_3^2 f_{12}^2}{f_1^2 f_4^2 f_6^2} = 12 \frac{f_2^{19} f_6^3}{f_1^8 f_4^8}.$$

By (2.7) we get

$$\sum_{n=0}^{\infty} \omega(3n+2) q^n = 12 \frac{f_2^{19} f_6^3}{f_4^8} \left(\frac{f_4^{28}}{f_2^{28} f_8^8} + 8q \frac{f_4^{16}}{f_2^{24}} + 16q^2 \frac{f_4^4 f_8^8}{f_2^{20}} \right). \quad (2.43)$$

If we extract the terms involving q^{2n+1} , divide by q and replace q^2 by q , we obtain

$$\sum_{n=0}^{\infty} \omega(6n+5) q^n = 96 \frac{f_2^8 f_3^3}{f_1^5}. \quad (2.44)$$

Comparing (2.44) with (2.35), we complete our proof. \square

Theorem 2.7. *For any nonnegative integer n , we have*

$$\omega(12n + 2) = 12B_3(6n).$$

Proof. If we extract the terms involving q^{2n} in (2.43) and then replace q^2 by q , we get

$$\sum_{n=0}^{\infty} \omega(6n + 2)q^n = 12 \left(\frac{f_2^{20} f_3^3}{f_1^9 f_4^8} + 16q \frac{f_3^3 f_4^8}{f_1 f_2^4} \right). \quad (2.45)$$

By (2.3) we obtain

$$\begin{aligned} \frac{f_3^3}{f_1^9} &= \left(\frac{f_4^6 f_6^3}{f_2^9 f_{12}^2} + 3q \frac{f_4^2 f_6 f_{12}^2}{f_2^7} \right)^3 \\ &= \left(\frac{f_4^{18} f_6^9}{f_2^{27} f_{12}^6} + 27q^2 \frac{f_4^{10} f_6^5 f_{12}^2}{f_2^{23}} \right) + 9q \left(\frac{f_4^{14} f_6^7}{f_2^{25} f_{12}^2} + 3q^2 \frac{f_4^6 f_6^3 f_{12}^6}{f_2^{21}} \right). \end{aligned} \quad (2.46)$$

Substituting (2.2) and (2.46) into (2.45), extracting the terms involving q^{2n} and then replacing q^2 by q , we deduce that

$$\sum_{n=0}^{\infty} \omega(12n + 2)q^n = 12 \left(\frac{f_2^{10} f_3^9}{f_1^7 f_6^6} + 16q \frac{f_2^7 f_6^3}{f_1^4} + 27q \frac{f_2^2 f_3^5 f_6^2}{f_1^3} \right). \quad (2.47)$$

Comparing (2.34) with (2.47), we deduce that $\omega(12n + 2) = 12B_3(6n)$. \square

Theorem 2.8. *For any nonnegative integer n , we have*

$$\omega(12n + 10) = 6B_3(6n + 4).$$

Proof. Extracting the terms involving q^{3n+1} in (2.42), dividing both sides by q and replacing q^3 by q , we get

$$\sum_{n=0}^{\infty} \omega(3n + 1)q^n = 6\varphi^3(q)\varphi(q^3)^2 \cdot \frac{f_2^2 f_3 f_{12}}{f_1 f_4 f_6}.$$

Substituting (2.36) into the above identity, we get

$$\sum_{n=0}^{\infty} \omega(3n + 1)q^n = 6 \frac{f_2^{15}}{f_1^6 f_4^6} \cdot \frac{f_6^{10}}{f_3^4 f_{12}^4} \cdot \frac{f_2^2 f_3 f_{12}}{f_1 f_4 f_6} = 6 \frac{f_2^{17} f_6^9}{f_1^7 f_3^3 f_4^7 f_{12}^3}. \quad (2.48)$$

Substituting (2.5) and (2.7) into (2.48), we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} \omega(3n + 1)q^n &= 6 \frac{f_2^{17} f_6^9}{f_4^7 f_{12}^3} \cdot \frac{1}{f_1^8} \cdot \frac{f_1}{f_3^3} \\ &= 6 \frac{f_2^{17} f_6^9}{f_4^7 f_{12}^3} \cdot \left(\frac{f_4^{28}}{f_2^{28} f_8^8} + 8q \frac{f_4^{16}}{f_2^{24}} + 16q^2 \frac{f_4^4 f_8^8}{f_2^{20}} \right) \left(\frac{f_2 f_4^2 f_{12}^2}{f_6^7} - q \frac{f_2^3 f_{12}^6}{f_4^2 f_6^9} \right). \end{aligned}$$

Extracting the term involving q^{2n+1} , dividing by q and then replacing q^2 by q , we get

$$\sum_{n=0}^{\infty} \omega(6n + 4)q^n = 6 \left(8 \frac{f_2^3 f_{12}^{11}}{f_1^8 f_6^8} - \frac{f_2^{19} f_3^3}{f_1^8 f_4^8} - 16q \frac{f_4^8 f_6^3}{f_2^5} \right). \quad (2.49)$$

By (2.3) we have

$$\frac{f_3^2}{f_1^6} = \left(\frac{f_4^6 f_6^3}{f_2^9 f_{12}^2} + 3q \frac{f_4^2 f_6 f_{12}^2}{f_2^7} \right)^2 = \frac{f_4^{12} f_6^6}{f_2^{18} f_{12}^4} + 6q \frac{f_4^8 f_6^4}{f_2^{16}} + 9q^2 \frac{f_4^4 f_6^2 f_{12}^4}{f_2^{14}}. \quad (2.50)$$

Substituting (2.7) and (2.50) into (2.49), extracting the terms involving q^{2n+1} , dividing by q and replacing q^2 by q , we obtain

$$\sum_{n=0}^{\infty} \omega(12n+10)q^n = 144 \frac{f_2^8 f_3^3}{f_1^5}.$$

Comparing this identity with (2.35), we deduce that $\omega(12n+10) = 6B_3(6n+4)$. \square

3. CONCLUDING REMARKS

Let $A_3^{(k)}(n)$ denote the number of partition k -tuples of n with 3-cores. In particular, $A_3^{(1)}(n) = a_3(n)$, $A_3^{(2)}(n) = A_3(n)$ appeared in the existing literature (see [2, 4, 12, 13]) and $A_3^{(3)}(n) = B_3(n)$ in this paper.

Again, let $\omega^{(k)}(n)$ denote the number of representations of a nonnegative integer n in the form

$$n = x_1^2 + \cdots + x_k^2 + 3(y_1^2 + \cdots + y_k^2), \quad x_i, y_i \in \mathbb{Z}, \quad i = 1, 2, \dots, k.$$

It is easy to see that the generating function of $A_3^{(k)}(n)$ and $\omega^{(k)}(n)$ are given by

$$\sum_{n=0}^{\infty} A_3^{(k)}(n)q^n = \frac{f_3^{3k}}{f_1^k}, \quad \text{and} \quad \sum_{n=0}^{\infty} \omega^{(k)}(n)q^n = \varphi^k(q)\varphi^k(q^3)$$

respectively.

From the existing papers and our work, we know many arithmetic identities about $A_3^{(k)}(n)$ for $k = 1, 2, 3$. Meanwhile, we have seen some relations between $A_3^{(k)}(n)$ and $\omega^{(k)}(n)$, such as

$$\begin{aligned} \omega^{(1)}(12n+4) &= 6A_3^{(1)}(n), \\ \omega^{(2)}(6n+5) &= 12A_3^{(2)}(2n+1), \\ \omega^{(3)}(6n+5) &= 4A_3^{(3)}(6n+4). \end{aligned}$$

Based on these facts and observations, we would like to ask the following two questions.

Question 1. Can we find some arithmetic identities involving $A_3^{(k)}(n)$ for all k ?

Question 2. Can we find some arithmetic relations between $A_3^{(k)}(n)$ and $\omega^{(k)}(n)$ for all k ?

To answer these questions, we believe that one may need to develop some new methods and ideas.

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